Proof of Jordan-von Neumann theorem for vector spaces over $\mathbb{R}$

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Abstract

The present work provides a proof of Jordan-von Neumann theorem for real vector spaces. The proof presented here is a somewhat more detailed, particular case of the complex treatment given by P. Jordan and J. v. Neumann in [1].

Index terms— Jordan-von Neumann Theorem, Vector spaces over $\mathbb{R}$, Parallelogram law

Definition 1 (Inner Product)**.** *Let E be a vector space over $\mathbb{R}$. A function $\langle .,. \rangle : E \times E \to \mathbb{R}$ is an* ***inner product*** *iff*

P1 $x \neq 0 \Rightarrow \langle x, x \rangle > 0, \forall x \in E$

P2 $\langle x, y \rangle = \langle y, x \rangle, \forall x, y \in E$

P3 $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \forall x, y, z \in E$

P4 $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle, \forall \lambda \in \mathbb{R}, \forall x, y \in E$

Definition 2. *A normed vector space $(E, \|.\|)$ satisfies the* ***parallelogram law*** *iff*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2), \forall x, y \in E$$

Theorem 3. *Let $(E, \langle .,. \rangle)$ be an inner product vector space and define $\|x\| = \sqrt{\langle x, x \rangle}$. Then, the normed vector space $(E, \|.\|)$ satisfies the parallelogram law.*

Proof. Given $x, y \in E$, by definition 1,

$$
\begin{aligned}
&\|x+y\|^2 + \|x-y\|^2 \\
&\overset{\|.\|}{=} \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&\overset{\text{P3}}{=} \langle x, x+y\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle + \langle -y, x-y\rangle \\
&\overset{\text{P2}}{=} \langle x+y, x\rangle + \langle x+y, y\rangle + \langle x-y, x\rangle + \langle x-y, -y\rangle \\
&\overset{\text{P3}}{=} \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle + \langle x, x\rangle + \langle -y, x\rangle + \langle x, -y\rangle + \langle -y, -y\rangle \\
&\overset{\text{P4}}{=} 2\langle x, x\rangle + \cancel{\langle y, x\rangle} + \langle x, y\rangle + \langle y, y\rangle - \cancel{\langle y, x\rangle} + \langle x, -y\rangle - \langle y, -y\rangle \\
&\overset{\text{P2}}{=} 2\langle x, x\rangle + \langle x, y\rangle + \langle y, y\rangle + \langle -y, x\rangle - \langle -y, y\rangle \\
&\overset{\text{P4}}{=} 2\langle x, x\rangle + \langle x, y\rangle + \langle y, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\
&\overset{\text{P2}}{=} 2\langle x, x\rangle + \cancel{\langle x, y\rangle} + 2\langle y, y\rangle - \cancel{\langle x, y\rangle} \\
&= 2(\langle x, x\rangle + \langle y, y\rangle) \\
&\overset{\|.\|}{=} 2(\|x\|^2 + \|y\|^2)
\end{aligned}
$$

□

Theorem 4. *Let $(E, \langle ., .\rangle)$ be an inner product vector space and define $\|x\| = \sqrt{\langle x, x\rangle}$. Then,*

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$$

Proof. Given $x, y \in E$, by definition 1,

$$
\begin{aligned}
&\|x+y\|^2 - \|x-y\|^2 \\
&\overset{\|.\|}{=} \langle x+y, x+y\rangle - \langle x-y, x-y\rangle \\
&\overset{\text{P3}}{=} \langle x, x+y\rangle + \langle y, x+y\rangle - \langle x, x-y\rangle - \langle -y, x-y\rangle \\
&\overset{\text{P2}}{=} \langle x+y, x\rangle + \langle x+y, y\rangle - \langle x-y, x\rangle - \langle x-y, -y\rangle \\
&\overset{\text{P3}}{=} \cancel{\langle x, x\rangle} + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle - \cancel{\langle x, x\rangle} - \langle -y, x\rangle - \langle x, -y\rangle - \langle -y, -y\rangle \\
&\overset{\text{P4}}{=} \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle + \langle y, x\rangle - \langle x, -y\rangle + \langle y, -y\rangle \\
&\overset{\text{P2}}{=} 3\langle x, y\rangle + \langle y, y\rangle - \langle -y, x\rangle + \langle -y, y\rangle \\
&\overset{\text{P4}}{=} 3\langle x, y\rangle + \cancel{\langle y, y\rangle} + \langle y, x\rangle - \cancel{\langle y, y\rangle} \\
&\overset{\text{P2}}{=} 4\langle x, y\rangle
\end{aligned}
$$

□

Theorem 5. *Let $(E, \|.\|)$ be a normed vector space which satisfies the parallelogram law, and define*

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

Then, $\langle ., .\rangle$ is an inner product.

Proof. To prove this we verify each property of definition 1.

P1 Given $x \in E$ such that $x \neq 0$,

$$\langle x, x\rangle = \frac{\|x+x\|^2 - \|x-x\|^2}{4} = \frac{\|2x\|^2 - \|0\|^2}{4} = \frac{\cancel{4}\,\|x\|^2}{\cancel{4}} = \|x\|^2 > 0$$

P2 Given $x, y, \in E$,

$$\begin{aligned}\langle x, y\rangle &= \frac{\|x+y\|^2 - \|x-y\|^2}{4} = \frac{\|x+y\|^2 - (|-1|\,\|x-y\|)^2}{4} \\ &= \frac{\|x+y\|^2 - \|(-1)(x-y)\|^2}{4} = \frac{\|y+x\|^2 - \|y-x\|^2}{4} = \langle y, x\rangle\end{aligned}$$

P3 Given $x, y, z \in E$, by the parallelogram law,

$$\|(x+z)+y\|^2 + \|(x+z)-y\|^2 = 2(\|x+z\|^2 + \|y\|^2) \tag{1}$$

$$\|(x-z)+y\|^2 + \|(x-z)-y\|^2 = 2(\|x-z\|^2 + \|y\|^2) \tag{2}$$

Subtracting equation 2 from equation 1,

$$\begin{aligned}&\|(x+z)+y\|^2 - \|(x-z)+y\|^2 + \|(x+z)-y\|^2 - \|(x-z)-y\|^2 \\ &\qquad = 2(\|x+z\|^2 + \|y\|^2) - 2(\|x-z\|^2 + \|y\|^2)\end{aligned}$$

$$\begin{aligned}&\Rightarrow (\|(x+y)+z\|^2 - \|(x+y)-z\|^2) + (\|(x-y)+z\|^2 - \|(x-y)-z\|^2) \\ &\qquad = 2(\|x+z\|^2 + \cancel{\|y\|^2} - \|x-z\|^2 - \cancel{\|y\|^2})\end{aligned}$$

$$\Rightarrow 4\langle x+y, z\rangle + 4\langle x-y, z\rangle = 8\langle x, z\rangle$$

$$\Rightarrow \langle x+y, z\rangle + \langle x-y, z\rangle = 2\langle x, z\rangle \tag{3}$$

Therefore, given $x', y', z' \in E$ (by equation 3),

$$\left\langle \left(\boxed{\frac{x'+y'}{2}}^{x}\right) + \left(\boxed{\frac{x'-y'}{2}}^{y}\right), z' \right\rangle + \left\langle \left(\frac{x'+y'}{2}\right)^{\boxed{x}} - \left(\frac{x'-y'}{2}\right)^{\boxed{y}}, z' \right\rangle = 2\left\langle \left(\frac{x'+y'}{2}\right)^{\boxed{x}}, z' \right\rangle$$

$$\overset{\text{P4}}{\Rightarrow} \left\langle \frac{x' + \cancel{y'} + x' - \cancel{y'}}{2}, z' \right\rangle + \left\langle \frac{\cancel{x'} + y' - \cancel{x'} + y'}{2}, z' \right\rangle = \langle x' + y', z' \rangle$$

$$\Rightarrow \left\langle \frac{\cancel{2}x'}{\cancel{2}}, z' \right\rangle + \left\langle \frac{\cancel{2}y'}{\cancel{2}}, z' \right\rangle = \langle x' + y', z' \rangle$$

$$\Rightarrow \langle x', z' \rangle + \langle y', z' \rangle = \langle x' + y', z' \rangle$$

$\boxed{\text{P4}}$ Given $x, y, \in E$, define $S = \{\lambda \in \mathbb{R} \mid \lambda \langle x, y \rangle = \langle \lambda x, y \rangle\}$. Let us proceed to prove that $S = \mathbb{R}$.

($\mathbb{Z} \subset S$) Clearly $1\langle x, y\rangle = \langle 1x, y \rangle = \langle x, y \rangle$ and thus $1 \in S$. Suppose that $\alpha, \beta \in S$, then

$$(\alpha \pm \beta)\langle x, y \rangle = \alpha \langle x, y \rangle \pm \beta \langle x, y \rangle \overset{\text{H.}}{=} \langle \alpha x, y \rangle \pm \langle \beta x, y \rangle$$
$$\overset{\text{P3}}{=} \langle \alpha x \pm \beta x, y \rangle = \langle (\alpha \pm \beta)x, y \rangle$$

Which means that $\alpha \pm \beta \in S$, and therefore $\mathbb{Z} \subset S$.

($\mathbb{Q} \subset S$) Suppose that $\alpha, \beta \in S$ and $\beta \neq 0$. Hence,

$$\alpha \langle x, y \rangle \overset{\text{H.}}{=} \langle \alpha x, y \rangle = \left\langle \frac{\beta}{\beta}\alpha x, y \right\rangle \overset{\text{H.}}{=} \beta \left\langle \frac{\alpha}{\beta} x, y \right\rangle$$
$$\Rightarrow \frac{\alpha}{\beta}\langle x, y \rangle = \left\langle \frac{\alpha}{\beta} x, y \right\rangle$$

Which means that $\frac{\alpha}{\beta} \in S$, and therefore $\mathbb{Q} \subset S$.

($\mathbb{R} \subset S$) Given $x, y \in E$, let f and g be real functions such that $f(\lambda) = \lambda \langle x, y \rangle$ and $g(\lambda) = \langle \lambda x, y \rangle$, for all $\lambda \in \mathbb{R}$. Considering the previous result, it is clear that $f(\lambda) = g(\lambda), \forall \lambda \in \mathbb{Q}$. Furthermore, both functions are continuous because f is linear and g is composition of continuous functions. This means that $f = g$, since two real continuous functions whose values coincide for every rational number must coincide for each irrational number as well. Therefore $\mathbb{R} \subset S$.

$\square$

References

[1] P. Jordan and J. v. Neumann; *On inner products in linear, metric spaces*; Annals of Mathematics, Vol. 36, No. 3; July, 1935.